Construction of asymmetric multivariate copulas

Eckhard Liebscher
University of Applied Sciences Merseburg
Department of Computer Sciences and Communication Systems
Geusaer Straße
06217 Merseburg
Germany

February 5, 2008

Abstract

In this paper we introduce two methods for the construction of asymmetric multivariate copulas. The first is connected with products of copulas. The second approach generalises the Archimedean copulas. The resulting copulas are asymmetric and may have more than two parameters in contrast to most of the parametric families of copulas described in the literature. We study the properties of the proposed families of copulas such as the dependence of two components (Kendall’s tau, tail dependence), marginal distributions and the generation of random variates.

key words: copula, Archimedean copula, tail dependence
MSC 2000: 62H20
1. Introduction

The aim of this paper is to construct multivariate families of asymmetric copulas. In the monographs by Nelsen (1999) and Joe (1997) the reader finds detailed accounts of the theory as well as surveys of commonly used copulas. Most of these copulas belong to Archimedean families with one or two parameters. So these copulas have a limited variety of shapes. Several authors have indicated that it is an open problem to find appropriate families of multivariate copulas (dimension greater than 2) with a flexible number of parameters which may be greater than two. Suitable families of copulas are also needed for parametric and semiparametric estimation methods for multivariate densities and distribution functions. Alponsi and Brigo (2005) describe a new construction method for asymmetric copulas based on periodic functions. A transformation method for two-dimensional copulas is discussed in Durrleman et al. (2000). In the present paper we introduce two universal methods for developing parametric families of copulas. The first one is connected with products of copulas and was proposed in a special case by Khoudraji (1995). The second approach generalises Archimedean families of copulas. The advantages of the families we propose are the following:

(i) The families are flexible in fitting data with a number of parameters which may be greater than two.

(ii) The one-dimensional and multivariate marginal distributions belong to the corresponding family of smaller dimension.

(iii) Methods are available for the generation of random variates.

(iv) The families are asymmetric and cover a wide range of dependencies.

The latter property is shown to hold for some specific families by means of the values of Kendall’s tau. Moreover, in the present paper we study tail dependence properties of the proposed copulas and provide sufficient conditions for positive quadrant dependence in the case of product copulas.

Appropriate families of copulas can be used for fitting multivariate densities to datasets. The parametric estimation problem of copulas is discussed in several papers (see e.g. Genest et al. (1995)). An efficient estimation method for parametric classes of copula densities is introduced and investigated in Chen et al. (2006). The asymptotic behaviour of two-stage estimation procedures is studied in Joe (2005). A
combination of kernel estimators for marginal densities and parametric estimators for the copula leads to the semiparametric estimators for multivariate densities examined in Liebscher (2005).

The paper is organised as follows: Section 2 introduces a construction principle for copulas based on products. Section 3 deals with generalised Archimedean copulas. The proofs are deferred to Section 4.

2. Products of copulas

2.1. Construction of the copulas

Let $X = (X^{(1)}, \ldots, X^{(d)})^T$ be a random vector having the joint distribution function $H$. We denote the marginal distribution function of $X^{(m)}$ by $F_m$, where $m \in \{1, \ldots, d\}$. According to Sklar’s theorem, we have

$$H(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d))$$

for $x_i \in \mathbb{R}$, where $C$ is the $d$-dimensional copula of $X$. Assuming the continuity of $F_1, \ldots, F_d$, the copula $C$ is uniquely determined. Concerning the definition and properties of copulas, we refer to the monograph by Nelsen (1999).

The following theorem provides a construction principle for copulas having the form of a product of copulas.

**Theorem 2.1.** Assume that $C_1, \ldots, C_k : [0, 1]^d \to [0, 1]$ are copulas. Let $g_{ji} : [0, 1] \to [0, 1]$ for $j = 1, \ldots, k, i = 1, \ldots, d$ be functions with the property that each of them is strictly increasing or is identically equal to 1. Suppose that $\prod_{j=1}^k g_{ji}(v) = v$ for $v \in [0, 1], i = 1, \ldots, d$, and $\lim_{v \to 0+} g_{ji}(v) = g_{ji}(0)$ for $j = 1, \ldots, k, i = 1, \ldots, d$. Then

$$\tilde{C}(u_1, \ldots, u_d) = \prod_{j=1}^k C_j(g_{j1}(u_1), \ldots, g_{jd}(u_d))$$

for $u_i \in [0, 1]$ is also a copula.

In general $\tilde{C}$ is obviously an asymmetric copula. In the special case of two-dimensional copulas, two factors ($k = 2$) and power functions $g_{ji}$, Khoudraji (1995)
already considered this construction technique. We refer also to the discussion on these copulas in Genest et al. (1998). Note that in addition, functions \( g_{ji} \equiv 1 \) are included in Theorem 2.1. It is not immediately apparent that generalisations to higher dimensions as given in Theorem 2.1 are reasonable. We have to check carefully that the function \( \tilde{C} \) defines a distribution function. Obviously, if \( C_1, \ldots, C_k \) are absolutely continuous, then copula \( \tilde{C} \) has a density. Under the assumptions of Theorem 2.1, we have

\[
\tilde{C}(1, u_2, \ldots, u_d) = \prod_{j=1}^{k} \tilde{C}_j(g_{j2}(u_2), \ldots, g_{j}(u_d)),
\]

where \( \tilde{C}_1, \ldots, \tilde{C}_k \) are appropriate \((d-1)\)-dimensional copulas since \( g_{j1}(1) = 1 \) for \( j = 1, \ldots, k \). This shows that the multivariate marginal distributions of \( \tilde{C} \) have the same product form as \( \tilde{C} \) itself.

If the assumptions of Theorem 2.1 are satisfied, then functions \( g_{ji} \) have the following properties:

(i) \( g_{ji}(1) = 1 \) and \( g_{ji}(0) = 0 \),

(ii) \( g_{ji} \) is continuous on \((0, 1] \).

(iii) If there are at least two functions \( g_{j1}, g_{j2} \) with \( 1 \leq j_1, j_2 \leq k \) which are not identically equal to 1, then \( g_{j1}(x) > x \) holds for \( x \in (0, 1), j = 1, \ldots, k \).

Now we give four alternatives for functions \( g_{ji} \) which are suitable for applications of Theorem 2.1:

(I) \( g_{ji}(v) = v^{\theta_{ji}} \) for \( j = 1, \ldots, k \), where \( \theta_{ji} \in [0, 1] \) and \( \sum_{j=1}^{k} \theta_{ji} = 1 \);

(II) \( g_{ji}(v) = v^{\theta_{ji}} e^{(v-1)\alpha_{ji}} \) for \( j = 1, \ldots, k \), where \( \sum_{j=1}^{k} \theta_{ji} = 1, \sum_{j=1}^{k} \alpha_{ji} = 0 \) and \( \theta_{ji} \in (0, 1), \alpha_{ji} \in (-\infty, 1), \theta_{ji} \geq -\alpha_{ji} \);

(III) \( g_{ji}(v) = v^{\theta_{ji}} (1 - e^{-\gamma_{ji}})^{-\alpha_{ji}} (1 - e^{-\gamma_{ji}})^{\alpha_{ji}} \) for \( j = 1, \ldots, k \), where \( \sum_{j=1}^{k} \theta_{ji} = 1, \sum_{j=1}^{k} \alpha_{ji} = 0 \) and \( \theta_{ji} \in (0, 1), \gamma_{ji} \in (0, +\infty), \alpha_{ji} \in (-\infty, 1), \alpha_{ji} \leq \theta_{ji} \);

(IV) \( g_{1i}(v) = \exp \left( \theta_i - \sqrt{\ln |v| + \theta_i^2} \right), g_{2i}(v) = v \exp \left( -\theta_i + \sqrt{\ln |v| + \theta_i^2} \right) \) for \( \theta_i \geq \frac{1}{2} \).

Here \( i \in \{1, \ldots, d\} \). The next example shows how to use the construction technique of Theorem 2.1.

**Example 1:** Suppose that \( C : [0, 1]^d \to [0, 1] \) is a copula, \( C_1 = C \) and \( C_2 \) is the
independent copula. Applying Theorem 2.1 with $g_{1i}(v) = v^{1-\theta_i}$, we obtain

$$\bar{C}(u_1, \ldots, u_d) = C(u_1^{1-\theta_1}, \ldots, u_d^{1-\theta_d}) \prod_{i=1}^{d} u_i^{\theta_i}$$

which is a copula for $\theta_i \in (0, 1)$. □

The proof of Theorem 2.1 is based on the following lemma:

**Lemma 2.1.** Let $(U^{(1)}_1, \ldots, U^{(k)}_d)^\top, \ldots, (U^{(1)}_1, \ldots, U^{(k)}_d)^\top$ be $k$ independent random vectors having distribution functions $H_1, \ldots, H_k$, on $[0, 1]^d$, respectively. Furthermore, let $g_{ji} : [0, 1] \rightarrow [0, 1]$ for $j = 1, \ldots, k$, $i = 1, \ldots, d$ be functions with the property that each of them is continuous and strictly increasing or is identically equal to 1. Suppose that $g_{ji}(1) = 1$ for $j = 1, \ldots, k$, $i = 1, \ldots, d$. Define $g_{ji}^{-1}(v) := 0$ for $v \leq g_{ji}(0)$. Then $u = (u_1, \ldots, u_d)^\top \mapsto H(u) = \prod_{j=1}^{k} H_j(g_{j1}(u_1), \ldots, g_{jd}(u_d))$ is the distribution function of $(\max_{j \in J_i} \{g_{ji}^{-1}(U^{(j)}_i)\})_{i=1,\ldots,d}$, $J_i = \{ j \in \{1, \ldots, k \} : g_{ji} \neq 1 \}$.

As it is seen from Lemma 2.1, there is a simple procedure for the generation of random variates with product copula $\bar{C}$ when generation procedures for copulas $C_j$ are available. Let $(U^{(1)}_1, \ldots, U^{(k)}_d)^\top, \ldots, (U^{(1)}_1, \ldots, U^{(k)}_d)^\top$ be $k$ independent random vectors having the distribution functions $C_1, \ldots, C_k$, respectively. Then $\bar{C}$ is the joint distribution of the random vector $(\max_{j \in J_i} \{g_{ji}^{-1}(U^{(j)}_i)\})_{i=1,\ldots,d}$ where $J_i$ is the set of indices $j$ with $g_{ji} \neq 1$.

**2.2. Properties of the proposed copulas**

In the sequel we clarify under which conditions two-dimensional product copulas

$$\bar{C}(u_1, u_2) = \prod_{j=1}^{k} C_j(g_{j1}(u_1), g_{j2}(u_2))$$

fulfill conditions of positive dependence. A distribution function $F$ has the TP$_2$ (totally positive of order 2) property if and only if

$$F(x_1, x_2)F(y_1, y_2) \geq F(x_1, y_2)F(y_1, x_2) \text{ for all } x_1 < y_1, x_2 < y_2$$

(see Joe (1997), p.23). Suppose that $(U, V)$ is a random vector having distribution function $C$ which is a two-dimensional copula. Then $V$ is left tail decreasing in $U$ if
for all $v \in [0,1]$, $u \mapsto P(V \leq v \mid U \leq u)$ is decreasing in $u$.

**Proposition 2.2.** Let $d = 2$. (i) If $C_1, \ldots, C_k$ are TP$_2$ distribution functions, then the product copula (1) with $g_{ji}$ as in Theorem 2.1 has also the TP$_2$ property. (ii) If $C_1, \ldots, C_k$ are left tail decreasing in one component, then copula (1) is also left tail decreasing in this component. (iii) Under the assumption of (i) or (ii), copula (1) is positive quadrant dependent, i.e. $C(u,v) \geq uv$.

If $f$ is TP$_2$, then the distribution is stochastically increasing and positive quadrant dependent. The dependence of two components can be described by Kendall’s tau given by $\tau = 4 \int_0^1 \int_0^1 C(u,v) \, dC(u,v) - 1$. Moreover, the upper/lower tail dependence coefficients are of interest:

$$
\lambda_U = \lim_{u \to 1^-} \frac{1 - 2u + C(u,u)}{1 - u}, \quad \lambda_L = \lim_{u \to 0^+} \frac{C(u,u)}{u}.
$$

Proposition 2.3 provides a nice property of product copulas:

**Proposition 2.3.** Let $\tau_0, \lambda_{U0}$ and $\lambda_{L0}$ be Kendall’s tau and the upper/lower tail dependence coefficient of the two-dimensional product copula (1). We denote Kendall’s tau and the upper/lower tail dependence coefficient of $C_j$ by $\tau_j, \lambda_{Uj}$ and $\lambda_{Lj}$ ($j = 1, \ldots, k$), respectively. Then

$$
\tau_0 \leq \min_{i=1,\ldots,k} \tau_i, \quad \lambda_{U0} \leq \min_{i=1,\ldots,k} \lambda_{Ui}, \quad \lambda_{L0} \leq \min_{i=1,\ldots,k} \lambda_{Li}.
$$

In the following subsections we discuss more specific examples. To simplify the formulas, we use only few special functions $g_{ji}$. The application of other functions $g_{ji}$ is analogous. We introduce

$$
\Pi(u) = \prod_{i=1}^d u_i, \quad M(u) = \min\{u_1, \ldots, u_d\}, \quad W(u) = \max\{u_1 + \ldots + u_d - d + 1, 0\}
$$

for all $u = (u_1, \ldots, u_d)^\top$. $M$ and $W$ are the upper and lower Fréchet bounds, respectively.
Choosing an Archimedean copula as the starting point for the construction, it is not easy to find multivariate \((d > 2)\) copula families approaching the lower Fréchet bound \(W\) for certain parameters. More precisely, by virtue of Corollary 4.6.3 by Nelsen (1999), \(C \succ W\) holds true for Archimedean copulas with strict and completely monotonic generators \((\succ\) means concordance ordering).

2.3. Clayton family

The multivariate version of the Clayton copula (sometimes called Cook-Johnson copula) reads as follows:

\[
C(u_1, \ldots, u_d) = \left(1 + \sum_{i=1}^{d} (u_i^{-\gamma} - 1)\right)^{-1/\gamma} \tag{2}
\]

(see Clayton (1978) for the bivariate case, and Nelsen (1999), p. 122 for the multivariate case). This copula is an Archimedean one with one parameter \(\gamma\). For simplicity suppose that \(\gamma \in [0, +\infty)\). The parameter \(\gamma\) can be negative but the admissible range of negative values shrinks as \(d\) increases. Using Theorem 2.1 in connection with Example 1, we obtain the family of copulas

\[
C^{(1)}(u_1, \ldots, u_d) = \prod_{i=1}^{d} u_i^{1-\theta_i} \left(1 + \sum_{i=1}^{d} (u_i^{-\gamma \theta_i} - 1)\right)^{-1/\gamma} \tag{3}
\]

which is an extension of family (2) with \(d + 1\) parameters \(\gamma \in [0, +\infty), \theta_1, \ldots, \theta_d \in [0, 1)\).

**Limiting cases of family (3):** (a) case \(\gamma \to 0\): \(C^{(1)} \to \Pi\). (b) case \(\gamma \to \infty, \theta_i \to 1\) for all \(i\): \(C^{(1)} \to M\).

Moreover, we can consider the product of two Clayton copulas with different parameters. In view of Theorem 2.1 with \(g_{ji}\) as in (I) or in (IV), we obtain other extensions:

\[
C^{(2)}(u_1, \ldots, u_d) = \left(1 + \sum_{i=1}^{d} (u_i^{-\gamma \theta_i} - 1)\right)^{-1/\gamma} \left(1 + \sum_{i=1}^{d} (u_i^{-\delta (1-\theta_i)} - 1)\right)^{-1/\delta}, \tag{4}
\]

\[
C^{(3)}(u_1, \ldots, u_d) = \left(1 + \sum_{i=1}^{d} \left(\exp\left(-\gamma \left(\theta_i - \sqrt{|\ln u_i| + \theta_i^2}\right)\right) - 1\right)\right)^{-1/\gamma} \left(1 + \sum_{i=1}^{d} \left(u_i^{-\delta} \exp\left(\delta \left(\theta_i - \sqrt{|\ln u_i| + \theta_i^2}\right)\right) - 1\right)\right)^{-1/\delta}. \tag{5}
\]
Copulas (4) and (5) have \(d + 2\) parameters: \(\gamma, \delta \in [0, +\infty), \theta_1, \ldots, \theta_d\), where \(\theta_i \in [0, 1)\) in case of copula (4), and \(\theta_i \in [\frac{1}{2}, \infty)\) in case of copula (5). Since the copula (2) has the TP\(_2\) property, the distribution functions (3) to (5) have this property, too.

**Limiting cases of family (4):** (a) case either \(\theta_i \rightarrow 0\) for all \(i\) or \(\theta_i \rightarrow 1\) for all \(i\): \(C^{(2)}\) approaches the Clayton copula in (2). (b) case either \(\gamma, \delta \rightarrow 0\) or \((d = 2, \theta_1 \rightarrow 0\) and \(\theta_2 \rightarrow 1\): \(C^{(2)} \rightarrow \Pi\). (c) case \(\theta_i = \theta_1\) for all \(i\) and \(\gamma, \delta \rightarrow \infty\): \(C^{(2)} \rightarrow M\).

**Limiting cases of family (5):** (a) case \(\theta_i \rightarrow \infty\) for all \(i\): \(C^{(3)}\) approaches the Clayton copula in (2). (b) case \(\gamma, \delta \rightarrow 0\): \(C^{(3)} \rightarrow \Pi\). (c) case \(\theta_i = \theta_1\) for all \(i\) and \(\gamma, \delta \rightarrow \infty\): \(C^{(3)} \rightarrow M\).

For families (3) to (5), the marginal distributions of dimension smaller than \(d\) belong also to the family (3) to (5), respectively. Studying the dependence structure of two components, we obtain the following results for Kendall’s tau \(\tau\):

**Table 1:** Kendall’s tau \(\tau\) in case of family (4) for various values of the parameters

<table>
<thead>
<tr>
<th>(\gamma)</th>
<th>(\delta)</th>
<th>(\theta_1)</th>
<th>(\theta_2)</th>
<th>Kendall’s tau</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>6</td>
<td>0.0</td>
<td>0.0</td>
<td>0.75</td>
</tr>
<tr>
<td>0.0</td>
<td>0.2</td>
<td>0.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.4</td>
<td>0.45</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.9</td>
<td>0.075</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.2</td>
<td>0.521406</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.4</td>
<td>0.424828</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.9</td>
<td>0.115877</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.4</td>
<td>0.379485</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.9</td>
<td>0.154278</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>0.9</td>
<td>0.214394</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\gamma)</th>
<th>(\delta)</th>
<th>(\theta_1)</th>
<th>(\theta_2)</th>
<th>Kendall’s tau</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>25</td>
<td>0.0</td>
<td>0.0</td>
<td>0.925965</td>
</tr>
<tr>
<td>0.0</td>
<td>0.2</td>
<td>0.740741</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.4</td>
<td>0.555556</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.9</td>
<td>0.0925925</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.2</td>
<td>0.689878</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.4</td>
<td>0.588440</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.9</td>
<td>0.190630</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.4</td>
<td>0.579587</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.9</td>
<td>0.287382</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>0.9</td>
<td>0.481449</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 2: Kendall’s tau τ in case of family (5) for various values of the parameters

<table>
<thead>
<tr>
<th>γ</th>
<th>δ</th>
<th>θ₁</th>
<th>θ₂</th>
<th>Kendall’s tau</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>6</td>
<td>0.5</td>
<td>0.5</td>
<td>0.240324</td>
</tr>
<tr>
<td>0.5</td>
<td>2</td>
<td>0.271514</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.488130</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>7</td>
<td>0.623877</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>0.550154</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>0.654175</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>20</td>
<td>0.620352</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>0.564917</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>20</td>
<td>0.681897</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>20</td>
<td>0.713969</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>25</td>
<td>0.5</td>
<td>0.5</td>
<td>0.474855</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>0.411192</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.660429</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>7</td>
<td>0.348398</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>0.700267</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>0.816933</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>20</td>
<td>0.329340</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>0.705130</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>20</td>
<td>0.845468</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>20</td>
<td>0.882889</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The values of Kendall’s tau were computed numerically using Maple. For example, it follows from the above tables that $C^{(2)}$ with $d = 3$ and parameters $γ = 0.6, δ = 6, θ₁ = 0, θ₂ = 0.2, θ₃ = 0.9$ has the marginal distributions $C^{(2)}(.,1)$ and $C^{(2)}(1,.)$ with $τ = 0.6$ and $τ = 0.075$, respectively, and is hence significantly asymmetric. The following figures show sectional views of the copula densities of families (4) and (5). Although Kendall’s tau is similar and parameters $γ$ and $δ$ are equal, the shape of the densities is significantly different.

Insert Figures 1 and 2 here

Therefore, the families (3) to (5) cover a large variety of dependencies.

2.4. The Gumbel family

The multivariate version of the Gumbel copula is defined by

$$C(u_1, \ldots, u_d) = \exp \left( - \left( \sum_{i=1}^{d} (-\ln u_i)^\gamma \right)^{1/\gamma} \right) \quad (6)$$

(see Gumbel (1960) for the bivariate case, and Nelsen (1999), p. 123 for the multivariate case) where the parameter is $\gamma \in [1, +\infty)$. An application of Theorem 2.1,
with \( C_1, C_2 \) chosen to be Gumbel copulas and \( g_{ji} \) as in (I), leads to the copula

\[
C^{(4)}(u_1, \ldots, u_d) = \exp \left( - \left( \sum_{i=1}^{d} (-\theta_i \ln u_i)^\gamma \right)^{1/\gamma} - \left( \sum_{i=1}^{d} (-(1 - \theta_i \ln u_i)^\delta \right)^{1/\delta} \right).
\]

Here, \( \gamma, \delta \in [1, +\infty), \theta_1, \ldots, \theta_d \in (0, 1] \) are the parameters. The marginal distributions of dimension smaller than \( d \) belong also to the family (6). The copula (7) has the TP\(_2\) property.

**Limiting cases of family (7):** (a) case \( \theta_i = \theta_1 \) for all \( i \) and \( \gamma \to \delta \): \( C^{(4)} \) approaches the Gumbel copula in (6). (b) case either \( \gamma, \delta \to 1 \) or \( d = 2, \theta_1 \to 0 \) and \( \theta_2 \to 1 \): \( C^{(4)} \to \Pi \). (c) case \( \theta_i = \theta_1 \) for all \( i \) and \( \gamma, \delta \to \infty \): \( C^{(4)} \to M \).

### 2.5. Frank family

Here we consider the two-dimensional case where \( C_1 \) and \( C_2 \) are two Frank copulas with parameters \( \gamma, \delta \in \mathbb{R} \). For \( g_{ji} \) as in (I), the resulting copula is given by

\[
C^{(5)}(u_1, u_2) = \frac{1}{\gamma \delta} \ln \left( 1 - (1 - e^{-\gamma})^{-1} \left( 1 - \exp(-\gamma u_1^{\theta_1}) \right) \left( 1 - \exp(-\gamma u_2^{\theta_2}) \right) \right) \\
\times \ln \left( 1 - (1 - e^{-\delta})^{-1} \left( 1 - \exp(-\delta u_1^{1-\theta_1}) \right) \left( 1 - \exp(-\delta u_2^{1-\theta_2}) \right) \right)
\]

(see Frank (1978), Nelsen (1986), Genest (1987)) with parameters \( \theta_i \in (0, 1), \gamma, \delta \in \mathbb{R} \). For this copula family, we can find combinations of parameters such that \( C^{(5)} \) approaches the lower Fréchet bound \( W \).

**Limiting cases:** (a) case \( \delta \to -\infty, \theta_1, \theta_2 \to 0 \): \( C^{(5)} \to W \). (b) case either \( \delta \to -\infty, \theta_1 \to 0, \theta_2 \to 1 \) or \( \gamma, \delta \to 0 \): \( C^{(5)} \to \Pi \). (c) case \( \theta_1 = \theta_2, \gamma, \delta \to \infty \): \( C^{(5)} \to M \).

### 2.6. Koehler-Symanowski family

Let us consider the Koehler-Symanowski copula

\[
C^{(6)}(u_1, \ldots, u_d) = \prod_{i=1}^{d-1} \prod_{j=i+1}^{d} \left( u_i^{1/\alpha_{ii}} + u_j^{1/\alpha_{jj}} - u_i^{1/\alpha_{ii}} u_j^{1/\alpha_{jj}} \right)^{\alpha_{ij}},
\]

with \( \alpha_{ij} \in \mathbb{R} \).
\[ \alpha_{i+} = \alpha_i + \sum_{j \neq i} \alpha_{ij}, \quad \alpha_{ij} = \alpha_{ji} \text{ for } i \neq j, \]
which is the copula of Koehler and Symanowski (1995) specialised to parameters with at most two indices. The parameters are given by \( \alpha_1, \ldots, \alpha_d > 0, \alpha_{ij} > 0 \) for \( i, j = 1 \ldots d, i < j \). By simple algebra,

\[
C^{(6)}(u_1, \ldots, u_d) = \prod_{i=1}^{d} u_i^{\theta_{ii}} \prod_{i=1}^{d-1} \prod_{j=i+1}^{d} \left( \left( u_i^{\theta_{ij}} \right)^{-1/\alpha_{ij}} + \left( u_j^{\theta_{ji}} \right)^{-1/\alpha_{ij}} - 1 \right)^{-\alpha_{ij}},
\]

where \( \theta_{ij} = \alpha_{ij}/\alpha_{i+} \) for \( i \neq j \), \( \theta_{ii} = \alpha_i/\alpha_{i+} \). Note that \( \sum_{j=1}^{d} \theta_{ij} = 1 \) for all \( i \), and \( \theta_{ij} \geq 0 \). Obviously, this copula is a product of an independent copula and Clayton copulas with parameters \( 1/\alpha_{ij} \) according to \( \bar{C} \) as in Theorem 2.1.

3. A generalisation of Archimedean copulas

3.1. Archimedean copulas

Archimedean copulas are widely used in applications. They are defined by

\[
C(u_1, \ldots, u_d) = \varphi^{-1}(\varphi(u_1) + \ldots + \varphi(u_d)) \quad \text{for } u_i \in [0, 1],
\]

where \( \varphi : [0, 1] \to [0, +\infty) \) is a strictly decreasing function with \( \lim_{t \to 0^+} \varphi(t) = +\infty, \varphi(1) = 0 \). This function \( C \) defines a copula if \( \varphi^{-1} \) is \( d \)-monotonic; i.e., \( (-1)^{k-1} \frac{d^k}{dt^k} \varphi^{-1}(t) \geq 0 \) for \( t \in (0, \infty), k = 1, \ldots, d \), see Nelsen (1999), p. 124. An equivalent characterisation of generators for Archimedean copulas may be found in McNeil and Nešlehová (2007). Archimedean copulas are symmetric; i.e., \( C \) is constant for all permutations of the arguments. These copulas can be rewritten in the form

\[
C(u_1, \ldots, u_d) = \tilde{\varphi}^{-1}(\tilde{\varphi}(u_1) \cdot \ldots \cdot \tilde{\varphi}(u_d)) \quad \text{for } u_i \in [0, 1] \quad (8)
\]

by means of a multiplicative generator \( \tilde{\varphi} : [0, 1] \to [0, 1], \tilde{\varphi}(t) = \exp(-\varphi(t)) \).
3.2. Generalisation

Let us replace the product \( \varphi(u_1) \cdots \varphi(u_d) \) in formula (8) by an average of products 
\( h_{j1}(\psi(u_1)) \cdots h_{jd}(\psi(u_d)) \) leading to 
\[
C(u) = \Psi \left( \frac{1}{m} \sum_{j=1}^{m} h_{j1}(\psi(u_1)) \cdots h_{jd}(\psi(u_d)) \right)
\] (9)
for \( u = (u_1, \ldots, u_d)^\top \in [0,1]^d \). Here \( \psi = \Psi^{-1} \) and \( \Psi, h_{jk} : [0,1] \to [0,1] \) are strictly increasing functions for \( j = 1, \ldots, m, k = 1, \ldots, d \). According to (9), \( \Psi^{-1}(C(u)) \) is an average of products of functions \( h_{j1}(\psi(.)), \) which can be regarded as an expansion of \( \Psi^{-1}(C(.)) \). The more summands are used in applications the better the approximation of the transformed copula can be. Function \( C \) of (9) represents a generalisation of Archimedean copulas being asymmetric in general. In the sequel we provide conditions on functions \( \Psi \) and \( h_{jk} \) ensuring \( C \) defined in (9) to be a copula. The following example shows that the copulas of the Frank family has the form (9) in the case \( m = 1 \).

**Example 2:** The multivariate copula of the Frank family is given by 
\[
C(u) = -\frac{1}{\gamma} \ln \left( 1 - (1 - e^{-\gamma})^{-d+1} \prod_{i=1}^{d} (1 - \exp(-\gamma u_i)) \right) \text{ for } u_i \in [0,1]
\] (see Nelsen (1999), p.123). This copula can be obtained from (9) by setting \( m = 1, \) 
\( h_{1k}(v) = v, \) \( \psi(v) = (1 - e^{-\gamma v}) (1 - e^{-\gamma})^{-1} \) and \( \Psi(t) = -\frac{1}{\gamma} \ln \left( 1 - (1 - e^{-\gamma})t \right) \).

**Theorem 3.1.** Assume that \( \Psi^{(d)} \) exist, \( \Psi'(u) > 0 \) and \( \Psi^{(i)}(u) \geq 0 \) for \( i = 2, \ldots, d, u \in [0,1] \). Let \( \Psi(0) = 0 \) and \( \Psi(1) = 1. \) Suppose that for each \( j \in \{1, \ldots, m\} \) and \( k \in \{1, \ldots, d\}, \) \( h_{jk} : [0,1] \to [0,1] \) is a differentiable and strictly increasing function with \( h_{jk}(0) = 0, h_{jk}(1) = 1, \) and 
\[
\frac{1}{m} \sum_{j=1}^{m} h_{jk}(x) = x \text{ for } k = 1, \ldots, d, x \in [0,1].
\] (10)
Then \( C \) defined in (9) is an absolutely continuous copula.

Under the assumptions of Theorem 3.1, the multivariate marginal distribution functions of \( C \) can be written in the form of equation (9).

We can choose functions \( h_{jk} \) as follows:
(I) \[ h_{jk}(x) = x^{\delta_{jk}} \] for \( j = 1, \ldots, m-1, x \in [0, 1], \quad h_{mk}(x) = mx^{m-1} \sum_{j=1}^{m-1} x^{\delta_{jk}} \] for \( x \in [0, 1] \)

with \( \delta_{jk} \in [1, 2], \sum_{j=1}^{m-1} \delta_{jk} \leq m, \)

(II) case \( m = 2, \)

\[ h_{1k}(x) = \frac{e^{a_k x} - 1}{e^{a_k} - 1}, \quad h_{2k}(x) = 2x - \frac{e^{a_k x} - 1}{e^{a_k} - 1} \] for \( x \in [0, 1] \)

with \( a_k \in (0, 1.59362], \)

(III) case \( m = 2, \)

\[ h_{1k}(x) = \frac{(a_k + 1)x}{1 + a_k x}, \quad h_{2k}(x) = 2x - \frac{(a_k + 1)x}{1 + a_k x} \] for \( x \in [0, 1] \)

with \( a_k \in (-0.5, 1]. \) In these case, \( k = 1, \ldots, d. \)

These functions \( h_{jk} \) cover a wide range of shapes. The following Table 3 provides some proposals for functions \( \Psi \) fulfilling \( \Psi(0) = 0, \Psi(1) = 1 \) and \( \Psi^{(i)}(u) \geq 0 \) for all \( i \geq 1. \)

Table 3: Proposals for functions \( \Psi \) and \( \psi \)

<table>
<thead>
<tr>
<th>parameter</th>
<th>( \Psi(t) )</th>
<th>( \psi(x) = \Psi^{-1}(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ( \gamma \in (0, +\infty) )</td>
<td>(-\frac{1}{\gamma} \ln (1 - (1 - e^{-\gamma})t))</td>
<td>(\frac{1 - e^{-\gamma t}}{1 - e^{-\gamma}})</td>
</tr>
<tr>
<td>2 ( \gamma \in (0, +\infty), \delta \in (1, +\infty) )</td>
<td>(\frac{(1 - t/\delta)^{-\gamma} - 1}{(1 - 1/\delta)^{-\gamma} - 1})</td>
<td>(\delta - \left(\delta^{-\gamma}(1 - x) + x(\delta - 1)^{-\gamma}\right)^{-\frac{1}{\gamma}})</td>
</tr>
<tr>
<td>3 ( \gamma \in (0, +\infty) )</td>
<td>(\frac{e^{\gamma t - 1}}{e^{\gamma} - 1})</td>
<td>(\frac{1}{\gamma} \ln (1 + x(e^{\gamma} - 1)))</td>
</tr>
<tr>
<td>4 ( \alpha \in (0, +\infty), \gamma \in (1, +\infty) )</td>
<td>(\exp\left(-\alpha(1 - t)^{1/\gamma}\right) - e^{-\alpha})</td>
<td>(1 - \left(-\frac{\ln(e^{-\alpha} + x - xe^{-\alpha})}{\alpha}\right)^{\gamma})</td>
</tr>
<tr>
<td>5 ( \alpha, \gamma \in (0, +\infty), \beta \in (1, +\infty) )</td>
<td>(\frac{(\beta - e^{\alpha(1-t)})^{-\gamma} - (\beta - e^{-\alpha})^{-\gamma}}{(\beta - 1)^{-\gamma} - (\beta - e^{-\alpha})^{-\gamma}})</td>
<td>(1 + \frac{1}{\alpha} \ln \left(\beta - ((\beta - e^{-\alpha})^{-\gamma} + x ((\beta - 1)^{-\gamma} - (\beta - e^{-\alpha} - 1)^{1/\gamma})\right))</td>
</tr>
<tr>
<td>6 ( \alpha \in (0, +\infty), \gamma \in (0, 1) )</td>
<td>(\frac{e^{\alpha/(1-t)\gamma} - e^{\alpha}}{e^{\alpha/(1-\gamma)} - e^{\alpha}})</td>
<td>(\frac{\ln(e^{-\alpha} + xe^{-\alpha}(1-\gamma) - xe^{-\alpha})}{\gamma \ln(e^{-\alpha} + xe^{-\alpha}(1-\gamma) - xe^{-\alpha})})</td>
</tr>
<tr>
<td>7 ( \delta \in [1, +\infty), \gamma \in (0, 1) )</td>
<td>(\frac{\delta^{\gamma} - (\delta - t)^{\gamma}}{\delta^{\gamma} - (\delta - 1)^{\gamma}})</td>
<td>(\delta - (\delta^{\gamma} - x\delta^{\gamma} + x(\delta - 1)^{\gamma})^{\frac{1}{\gamma}})</td>
</tr>
</tbody>
</table>
The first line of Table 2 gives the function $\Psi$ corresponding to Frank copula. The property $\Psi^{(i)}(t) \geq 0$ for $i \geq 1$ can be verified using that $t \mapsto t^{-\gamma}$ with $\gamma > 0$, $t \mapsto e^{-\alpha t}$ with $\alpha > 0$ and $t \mapsto e^{\alpha t}$ with $\alpha > 0$ are completely monotonic, and $t \mapsto t^{1/\gamma}$ has a completely monotonic derivative for $\gamma > 1$. Furthermore, it can easily be seen that if $\delta \geq 1$ and $g$ is a completely monotonic function on $[0, \infty)$, then all derivatives (including the function itself) of the functions $x \mapsto g(\delta - x)$, $x \mapsto g(1 - \delta^{-1}x)$ are positive on $[0, 1]$. Thus all functions $\Psi$ of the table can be used for applications of Theorem 3.1.

Example 3: Generalised Frank copula with $h_{jk}$ as in case (I):

$$\Psi(t) = -\frac{1}{\gamma} \ln \left(1 - (1 - e^{-\gamma})t\right), \quad \psi(x) = \Psi^{-1}(x) = \frac{1 - e^{-\gamma x}}{1 - e^{-\gamma}},$$

$$h_{1k}(x) = x^{\delta_k}, \quad h_{2k}(x) = 2x - x^{\delta_k}, \quad \text{where } \gamma \in (0, +\infty), \delta_k \in [1, 2], k = 1, \ldots, d.$$ 

With these settings and $m = 2$, we consider the function $C$ given in (9). As explained above, we have $\Psi^{(k)}(u) \geq 0$ for $k \geq 1$, and $h_{1k}'(v) > 0$. Therefore by Theorem 3.1, $C$ is a copula with parameters $\gamma, \delta_1, \ldots, \delta_d$. If $\delta_k = 1$ for all $k$ except one, then the resulting $C$ coincides with the usual Frank copula. □

Limiting cases of Example 3: (a) case $\gamma \to 0, \delta_j \to 1$ for all $j$ except one: $C \to \Pi$. (b) case $\gamma \to \infty, \delta_j \to 1$ for all $j$ except one: $C^{(4)} \to M$.

The algorithm from the paper by Marshall and Olkin (1988) for generating random vectors with a given Archimedean copula which can be found explicitly in Frees and Valdez (1998, p. 12), is developed further in the next proposition. This statement provides a method for generating random variates with distribution (9).

Proposition 3.2. Let $Y_{11}, \ldots, Y_{d1}, Y_{12}, \ldots, Y_{d2}, \ldots$ be random variables which are uniformly distributed on $[0, 1]$, and $Z_1, Z_2, \ldots$ be random variables which are uniformly distributed on $\{1, \ldots, m\}$. Let the conditional distribution of $K \mid W$ be a Poisson distribution with parameter $W$. Assume that $\lim_{t \to 0} t^{-1} \Psi(t) > 0$ and the random variable $W$ has a density $f_W$ on $[0, +\infty)$ such that the function $\Lambda$ with $\Lambda(t) = (1 - t)^{-1} \Psi(1 - t)$ for $t \in [0, 1]$ is the Laplace transform of $f_W$ on $[0, 1]$. Suppose that $(W, K), Y_{11}, \ldots, Y_{d1}, Y_{12}, \ldots, Y_{d2}, \ldots, Z_1, Z_2, \ldots$ are independent random variables ($W$ and $K$ are dependent). Define $\tilde{Y}_{ik} = \Psi \left( h_{ji}^{-1}(Y_{ik}) \right)$ if $Z_k = j$. Then the random vector $(\max_{k=1,\ldots,K+1} \tilde{Y}_{1k}, \max_{k=1,\ldots,K+1} \tilde{Y}_{2k}, \ldots, \max_{k=1,\ldots,K+1} \tilde{Y}_{dk})^\top$ has distrib-
ution function $C$ according to (9).

The following figures show the shapes of copula densities of the generalised Frank copula as explained in Example 3 but with $h_{jk}$ as in case (III). These densities feature a large variety of shapes and the property of asymmetry.

Insert Figures 3 to 5 here

Next we provide computed values of Kendall’s tau of the preceding copula for various values of the parameters. This table shows that the family of generalised Frank copulas describe a wide range of dependencies:

Table 4: Kendall’s $\tau$ of copulas of the generalised Frank family

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>Kendall’s $\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>-0.4</td>
<td>0.95</td>
<td>0.116571507</td>
</tr>
<tr>
<td>0.5</td>
<td>0.95</td>
<td>0.197612659</td>
<td></td>
</tr>
<tr>
<td>0.95</td>
<td>0.95</td>
<td>0.218418821</td>
<td></td>
</tr>
<tr>
<td>-0.4</td>
<td>0.5</td>
<td>0.133718185</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>-0.4</td>
<td>0.95</td>
<td>0.279952481</td>
</tr>
<tr>
<td>3</td>
<td>-0.4</td>
<td>0.95</td>
<td>0.326483422</td>
</tr>
<tr>
<td>0.95</td>
<td>0.95</td>
<td>0.337663411</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>Kendall’s $\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>-0.4</td>
<td>0.5</td>
<td>0.289729160</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.319449883</td>
<td></td>
</tr>
<tr>
<td>-0.4</td>
<td>-0.4</td>
<td>0.333620140</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>-0.4</td>
<td>0.95</td>
<td>0.662061599</td>
</tr>
<tr>
<td>0.5</td>
<td>0.95</td>
<td>0.668266198</td>
<td></td>
</tr>
<tr>
<td>0.95</td>
<td>0.95</td>
<td>0.669668451</td>
<td></td>
</tr>
<tr>
<td>-0.4</td>
<td>0.5</td>
<td>0.663362087</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.667374542</td>
<td></td>
</tr>
<tr>
<td>-0.4</td>
<td>-0.4</td>
<td>0.669629621</td>
<td></td>
</tr>
</tbody>
</table>

This table shows that for large values of $\gamma$, Kendall’s $\tau$ is quite insensitive to parameters $a_1$ and $a_2$.

Finally, we consider the upper and lower tail dependence coefficients in case $d = 2$. An extensive discussion on tail dependence coefficients for Archimedean copulas can be found in Charpentier and Segers (2007). Suppose that $\psi$ is continuously
differentiable on $[0, 1]$. Then we can derive

$$\lambda_U = 2 - \lim_{u \to 1^-} \frac{d}{du} C(u, u)$$

$$= 2 - \frac{1}{m} \lim_{u \to 1^-} \psi' \left( \frac{1}{m} \sum_{k=1}^{m} h'_{k1}(\psi(u))h'_{k2}(\psi(u)) \psi'(u) \right)$$

$$\sum_{j=1}^{m} \left( h'_{j1}(\psi(u))h'_{j2}(\psi(u)) + h_{j1}(\psi(u))h'_{j2}(\psi(u)) \right)$$

$$= 2 - \frac{1}{m} \lim_{u \to 1^-} \psi' \left( \frac{1}{m} \sum_{k=1}^{m} h_k(t)k(t) \right) \psi'(t)^{-1}.$$  

Obviously, $\lim_{t \to 1^-} \psi'(t) < +\infty$ implies $\lambda_U = 0$. Hence the copula of Example 3 has no upper tail dependence. Analogously,

$$\lambda_L = \lim_{u \to 0^+} \frac{d}{du} C(u, u) = 0$$

provided that $\lim_{t \to 0^+} \psi' \left( \frac{1}{m} \sum_{k=1}^{m} h_k(t)k(t) \right) \psi'(t)^{-1} < \infty$.

In the following example, the copula is upper tail dependent.

**Example 4:** $\Psi(t) = (\exp(-\alpha(1 - t)^{1/\gamma}) - e^{-\alpha}) (1 - e^{-\alpha})^{-1}$ according to line 4 in Table 3. Here we have $\lim_{t \to 1^-} \psi'(t) = +\infty$. Then

$$\lambda_U = 2 - 2 \lim_{t \to 1^-} \psi' \left( \frac{1}{2} (h'_{11}(t)h'_{12}(t) + h_{21}(t)h'_{22}(t)) \right) \psi'(t)^{-1}$$

$$= 2 - 2 \lim_{t \to 1^-} \left( 1 - \frac{1}{2} \frac{(h'_{11}(t)h'_{12}(t) + h_{21}(t)h'_{22}(t))}{1 - t} \right)^{1/2}$$

$$= 2 - 2 \lim_{t \to 1^-} \left( \frac{1}{2} (h_{11}(t)h'_{12}(t) + h_{11}(t)h'_{12}(t) + h_{21}(t)h'_{22}(t) + h_{21}(t)h'_{22}(t))^{1/2} \right)^{1/2}$$

$$= 2 - 2^{\frac{3}{2}}.$$

4. Proofs

**Proof of Lemma 2.1:** Observe that for $u, v \in [0, 1]$, $v \leq g_{ji}(u)$ is equivalent to $g_{ji}^{-1}(v) \leq u$. Now we deduce

$$\prod_{j=1}^{k} H_j(g_{j1}(u_1), \ldots, g_{jd}(u_d)) = \mathbb{P} \left\{ U_1^{(j)} \leq g_{j1}(u_1), \ldots, U_d^{(j)} \leq g_{jd}(u_d) \text{ for } j = 1, \ldots, k \right\}$$

$$= \mathbb{P} \left\{ \max_{j=1, \ldots, k} \{ g_{j1}^{-1}(U_1^{(j)}) \} \leq u_1, \ldots, \max_{j=1, \ldots, k} \{ g_{jd}^{-1}(U_d^{(j)}) \} \leq u_d \right\}.$$
This identity implies the lemma. □

PROOF OF THEOREM 2.1: An application of Lemma 2.1 shows that $\tilde{C}(u)$ is a distribution function. Obviously, $g_{ji}(1) = 1$. Further we obtain

$$
\tilde{C}(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_d) = 0,
\tilde{C}(1, \ldots, 1, v, 1_{i+1}, \ldots, 1_d) = \prod_{j=1}^{k} g_{ji}(v) = v, \quad \text{for } i = 1 \ldots d.
$$

Therefore, $\tilde{C}(u)$ is a copula. □

PROOF OF PROPOSITION 2.2: a) Obviously, $C_j(g_{j1}(.), g_{j2}(.))$ satisfies the TP$_2$ condition since $g_{j1}$ and $g_{j2}$ are strictly increasing or one of them is equal to 1 ($j = 1, \ldots, d$). This implies immediately the validity of assertion a).
b) We have to show that $u \mapsto u^{-1}\tilde{C}(u,v)$ is decreasing for all $v$. Let $v \in [0,1]$ be fixed, and $K(v) = \prod_{j:1 \leq j \leq k, g_{j1} \equiv 1} g_{j2}(v)$. Observe that

$$
\frac{\tilde{C}(u,v)}{u} = K(v) \prod_{j:1 \leq j \leq k, g_{j1} \neq 1} \frac{C_j(g_{j1}(u), g_{j2}(v))}{g_{j1}(u)}.
$$

Each factor of the product determines a decreasing function in $u$. This leads to assertion b). Assertion c) is a consequence of Theorem 2.3 in Joe (1997). □

PROOF OF PROPOSITION 2.3: Let $j \in \{1, \ldots, k\}$ be arbitrary. Obviously,

$$
\prod_{i=1}^{k} C_i(g_{i1}(u), g_{i2}(v)) \leq C_j(g_{j1}(u), g_{j2}(v)) \quad \text{for } u, v \in [0,1],
$$

and $\tilde{C} := C_j(g_{j1}(.), g_{j2}(.))$ is a distribution function. Let $\prec$ be the symbol for concordance ordering. Hence $\tilde{C} \prec \tilde{C}$ holds which implies $\tau(\tilde{C}) \leq \tau(\tilde{C}) = \tau(C_j)$, see Joe (1997), p. 37. Similar inequalities hold true for the tail dependence coefficients. □

PROOF OF THEOREM 3.1: Observe that, by (10),

$$
C(u_1, 1, \ldots, 1) = \Psi \left( \frac{1}{m} \sum_{j=1}^{m} h_{j1}(\psi(u_1)) \right) = u_1,
C(0, u_2, \ldots, u_d) = \Psi(0) = 0.
$$

Next we show that $\frac{\partial^d}{\partial u_1 \partial u_2 \ldots \partial u_d} C(u) \geq 0$ for all $u$; i.e., the copula has a density.
Let $\tilde{\Psi}(\cdot) = \Psi(\cdot/m)$. By induction, we can prove that

$$
\frac{\partial^k}{\partial u_1 \partial u_2 \ldots \partial u_k} C(u) = \sum_{\nu=1}^{k} \tilde{\Psi}^{(\nu)} \left( \sum_{j=1}^{m} h_{j1}(\psi(u_1)) \ldots h_{jd}(\psi(u_d)) \right) \cdot 
\sum_{M_1 \cup \ldots \cup M_{\mu} \text{ is a decomposition of } \{1, \ldots, k\}} a_{\nu k}(M_1, \ldots, M_{\mu})
$$

(11)

for $k = 1, \ldots, d$. The notation $\frac{\partial^m}{\partial u_{i_1} \ldots \partial u_{i_m}}$ means $\frac{\partial^i}{\partial u_{i_1} \ldots \partial u_{i_i}}$, where $M = \{i_1, \ldots, i_i\}$. In formula (11), $a_{\nu k}(M_1, \ldots, M_{\mu})$ denotes an integer depending on $k, \nu, M_1, \ldots, M_{\mu}$. Obviously, (11) is true for $k = 1$. Differentiating the both sides of (11) w.r.t. $u_k+1$, the right hand side gets the form of the right hand side of (11) with $k$ replaced by $k + 1$. Therefore, by induction, we obtain that (11) is valid for all $1 \leq k \leq d$.

Since $\frac{\partial^m}{\partial u_{i_1} \ldots \partial u_{i_m}} \left( \sum_{j=1}^{m} h_{j1}(\psi(u_1)) \ldots h_{jd}(\psi(u_d)) \right) \geq 0$ for different $i_1, \ldots, i_m$ by assumption, we have $\frac{\partial^k}{\partial u_{M}} \left( \sum_{j=1}^{m} h_{j1}(\psi(u_1)) \ldots h_{jd}(\psi(u_d)) \right) \geq 0$ for $M \subset \{1, \ldots, k\}$.

Remember that $\tilde{\Psi}^{(\nu)}(u) \geq 0$, by assumption. Thus, the proof of this theorem is complete. □

**Proof of Proposition 3.2:** We derive

$$
\mathbb{P} \left\{ \max_{k=1}^{K+1} \tilde{Y}_{1k} \leq x_1, \ldots, \max_{k=1}^{K+1} \tilde{Y}_{dk} \leq x_d \right\}

= \sum_{\kappa=0}^{\infty} \int_{0}^{\infty} \mathbb{P} \left\{ \max_{k=1}^{\kappa+1} \tilde{Y}_{1k} \leq x_1, \ldots, \max_{k=1}^{\kappa+1} \tilde{Y}_{dk} \leq x_d, K = \kappa \mid W = w \right\} f_W(w) \, dw

= \int_{0}^{\infty} \sum_{\kappa=0}^{\infty} \frac{w^{\kappa}}{\kappa!} e^{-w} \left( \mathbb{P} \left\{ \tilde{Y}_{11} \leq x_1, \ldots, \tilde{Y}_{d1} \leq x_d \right\} \right)^{\kappa+1} f_W(w) \, dw.
$$

Further

$$
\mathbb{P} \left\{ \tilde{Y}_{11} \leq x_1, \ldots, \tilde{Y}_{d1} \leq x_d \right\}

= \sum_{j=1}^{m} \mathbb{P} \left\{ \Psi(h_{j1}^{-1}(Y_{11})) \leq x_1, \ldots, \Psi(h_{jd}^{-1}(Y_{d1})) \leq x_d \mid Z_1 = j \right\} \frac{1}{m}

= \sum_{j=1}^{m} \mathbb{P} \left\{ Y_{11} \leq h_{j1}(\psi(x_1)), \ldots, Y_{d1} \leq h_{jd}(\psi(x_d)) \right\} \frac{1}{m}

= \frac{1}{m} \sum_{j=1}^{m} h_{j1}(\psi(x_1)) \ldots h_{jd}(\psi(x_d)) =: \tilde{h}(x_1, \ldots, x_d) \text{ for } x_i \in [0, 1].
$$

Hence
\[ \mathbb{P} \left\{ \max_{k=1..K+1} \tilde{Y}_{1k} \leq x_1, \ldots, \max_{k=1..K+1} \tilde{Y}_{dk} \leq x_d \right\} \]
\[ = \int_0^\infty \sum_{k=0}^\infty \frac{w^k}{k!} e^{-w} \tilde{h}(x_1, \ldots, x_d)^{k+1} f_W(w) \, dw \]
\[ = \tilde{h}(x_1, \ldots, x_d) \int_0^\infty \exp \left( -w(1 - \tilde{h}(x_1, \ldots, x_d)) \right) f_W(w) \, dw \]
\[ = \tilde{h}(x_1, \ldots, x_d) \Lambda(1 - \tilde{h}(x_1, \ldots, x_d)) \]
\[ = \Psi(\tilde{h}(x_1, \ldots, x_d)), \]

where \( \Lambda \) is the Laplace transform of \( f_W \) on \([0, 1]\), and \( \Lambda(t) = (1 - t)^{-1} \Psi(1 - t) \) for \( t \in [0, 1] \). □

5. Acknowledgement

The author is grateful to an associate editor and to two referees for careful reading of the manuscript and for several valuable suggestions.

6. References


Figure 1: sectional view ($u_2 = 0.2$ solid line, $0.4$ dotted line, $0.6$ dashed line, $0.8$ dash-dot line) of the density of copula (4) with $d = 2, \gamma = 0.4, \delta = 3, \theta_1 = 0.04, \theta_2 = 0.525; \tau = 0.28347$
Figure 2: sectional view ($u_2 = 0.2$ solid line, $0.4$ dotted line, $0.6$ dashed line, $0.8$ dash-dot line) of the density of copula (5) with $d = 2, \gamma = 0.4, \delta = 3, \theta_1 = 0.8, \theta_2 = 2.7; \tau = 0.28658$
Figure 3: copula density of Frank copula with $\gamma = 4$
Figure 4: copula density of generalised Frank copula with $\gamma = 1$, $h_{jk}$ according to (3) with $a_1 = -0.5, a_2 = 0.8$
Figure 5: copula density of generalised Frank copula with $\gamma = 0.3$,

\[ h_{11}(x) = (x^a + (1-x)^a)^{1/a} - 1 + x,\ h_{12}(x) = \frac{(b+1)x^4}{x^4 + b},\ h_{2k}(x) = 2x - h_{1k}(x) \text{ with } a = 3, b = 0.1 \]