ADAPTIVE ALGEBRAIC RECONSTRUCTION OF CT IMAGES IN COMPARISON TO FREQUENCY DOMAIN METHODS

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Abstract

This paper presents a new algebraic algorithm for the reconstruction of computer tomographic (CT) images. Since the reconstruction is an ill posed problem a least squares approach with adaptively weighted linear shift invariant regularization functions is used. If the weighting functions are properly matched to the regularization functions the adaptive algorithm simultaneously leads to proper noise suppression and enhanced resolution of discontinuities. Simulations demonstrate the reconstruction capabilities of the new algorithm in comparison to filtered backprojection.

1. Statement of the Problem

Fig. 1 shows the data collection process in CT using parallel beam projections. It can be described as the application of a twodimensional linear shift variant filter [1] to the original image. Basically fan beam projections lead to the same considerations [1]. The generation of the measured data $g(t, \theta)$ can be described using a line integral. In x-ray-CT the noise $n(t, \theta)$ is Poisson distributed [2].

$$g(t, \theta) = \int \int s(x, y) \cdot \delta(x \cdot \cos \theta + y \cdot \sin \theta - t) \cdot dx dy + n(t, \theta)$$

(1)

Using a different point of view eq. 2 defines the linear shift variant point spread function of the twodimensional linear system which generates the measured data.

$$g(t, \theta) = \int \int s(x, y) \cdot h(x, y, t, \theta) \cdot dx dy + n(t, \theta)$$

$$h(x, y, t, \theta) = \delta(x \cdot \cos \theta + y \cdot \sin \theta - t)$$

(2)
The discrete equivalent of eq. 2 can be written using components:

\[
g(i_d, i_\theta) = \sum_{i_x} \sum_{i_y} g(i_x, i_y) \cdot h(i_x, i_y, i_d, i_\theta) + n(i_d, i_\theta)
\]  \hspace{1cm} (3)

or vector matrix notation:

\[
g = H \cdot s + n
\]  \hspace{1cm} (4)

**Figure 1: Parallel beam CT-scanner**

2. **Regularized Algebraic Reconstruction**

Since \( H \) represents a low pass system and the measured data \( g \) are noisy the reconstruction of \( s \) is an ill posed problem [3]. The quadratic error term \( E_Q \) includes a group of linear shift invariant regularization functions which are represented by the block Toeplitz matrices \( F_1^{-1}, F_2^{-1}, \ldots, F_M^{-1} \). Since we intend to develop an adaptive algorithm we introduce the inverse weighting matrices \( V_1^{-1}, V_2^{-1}, \ldots, V_M^{-1} \) and \( A^{-1} \).

\[
E_Q = (g - H\hat{s})^T A^{-1}(g - H\hat{s}) + \hat{s}^T \left( F_1^T V_1^{-1} F_1 + F_2^T V_2^{-1} F_2 + \ldots + F_M^T V_M^{-1} F_M \right) \hat{s}
\]  \hspace{1cm} (5)

The solution vector \( \hat{s} \) is found by minimization of the error \( E_Q \). Setting the derivative of \( E_Q \) with respect to the solution vector to zero leads to a linear system of equations.
\[
\frac{\partial E_Q}{\partial \hat{s}} = 0 \quad \Rightarrow \\
\hat{s} = (H^T A^{-1}H + F_1^T V_1^{-1}F_1 + F_2^T V_2^{-1}F_2 + \ldots + F_M^T V_M^{-1}F_M)^{-1} H^T A^{-1} \mathbf{g}
\]

\( V_1, V_2, \ldots, V_M \) and \( A \) are optimized by minimization of the expectation \( D_Q \) of the squared distance between the original signal \( \mathbf{s} \) and the estimated signal \( \hat{s} \).

\[
D_Q = \langle (\hat{s} - \mathbf{s})^T (\hat{s} - \mathbf{s}) \rangle 
\]

\[
\frac{\partial D_Q}{\partial V_m} = 0 \quad \forall \ m \quad \frac{\partial D_Q}{\partial A} = 0
\]

Straightforward application of matrix differentiation rules and matrix algebra leads to \( M+1 \) identical matrix equations so that there is an infinite number of solutions.

\[
\left( F_1^T V_{1opt}^{-1} F_1 + F_2^T V_{2opt}^{-1} F_2 + \ldots + F_M^T V_{Mopt}^{-1} F_M \right) \langle \mathbf{s} \mathbf{s}^T \rangle \cdot H^T = H^T A_{opt}^{-1} \langle \mathbf{n} \mathbf{n}^T \rangle
\]

Since each solution minimizes \( D_Q \) we are free to select the simplest one:

\[
A_{opt} = \langle \mathbf{n} \mathbf{n}^T \rangle
\]

\[
F_m^T V_{mopt}^{-1} F_m \langle \mathbf{s} \mathbf{s}^T \rangle H^T = \mu_m H^T A_{opt}^{-1} \langle \mathbf{n} \mathbf{n}^T \rangle \quad \forall \ m
\]

\[
\Rightarrow V_{mopt} = \mu_m^{-1} F_m \langle \mathbf{s} \mathbf{s}^T \rangle F_m \quad \forall \ m
\]

where:

\[
\mu_1 + \mu_2 + \ldots + \mu_M = 1 \quad \land \quad 0 \leq \mu_m \leq 1 \quad \forall \ m
\]

The matrices \( V_{1opt}, V_{2opt}, \ldots, V_{Mopt} \) and \( A_{opt} \) are determined by the autocorrelation matrices, in other words, by the statistics of the original signal and the noise. If they are inserted into eq. 7 the resulting estimate is similar to the estimate provided by a shift variant Wiener Filter [3,4]. If the signal and the noise are Gaussian distributed it is also similar to a Maximum a Posteriori estimate [3].

There are two reasons why this algorithm is not applicable to most practical problems. First the autocorrelation matrices must be known very accurately. Usually this is no problem for the noise. But a reliable estimate of the autocorrelation matrix of the original signal is hardly available. Second the size of the matrices is prohibitively large, especially for multidimensional signals. The autocorrelation matrix of an image which consists of not more than 100-100 pixels contains 10,000-10,000 coefficients!
3 Suboptimal Diagonal Weighting Matrices

In order to derive a suboptimal but applicable algorithm we do not longer search for optimal matrices in the general form but for optimal diagonal matrices.

\[ E_Q = (g - H\hat{s})^T A_D^{-1} (g - H\hat{s}) + s^T \left( F_1^T V_{D1}^{-1} F_1 + F_2^T V_{D2}^{-1} F_2 + \ldots + F_M^T V_{DM}^{-1} F_M \right) \hat{s} \]  
\[ \hat{s} = \left( H^T A_D^{-1} H + F_1^T V_{D1}^{-1} F_1 + F_2^T V_{D2}^{-1} F_2 + \ldots + F_M^T V_{DM}^{-1} F_M \right)^{-1} H^T A_D^{-1} g \] 

\( V_{D1}, V_{D2}, \ldots, V_{DM} \) and \( A_D \) are again optimized by minimization of \( D_Q \) (eq. 8).

\[ \frac{\partial D_Q}{\partial v_{Dm_{ii}}} = 0 \quad \forall \ m, i \qquad \frac{\partial D_Q}{\partial a_{D_{kk}}} = 0 \quad \forall \ k \]  

The simplest solution is:

\[ a_{D_{kkopt}} = \left( \frac{nn^T}{kk} \right) \Rightarrow v_{Dm_{iiopt}} = \mu_m^{-1} f_m \left( ss^T \right) F_m^T \] 

We only take into account the diagonal elements of the previously determined general weighting matrices. Obviously \( V_{Dmopt} \) and \( A_{Dopt} \) are the diagonal matrices which are closest to \( V_{mopt} \) and \( A_{opt} \). If the noise is white the optimal diagonal matrix \( A_D \) equals the general matrix \( A \). The local variances of the Poisson distributed noise which can be estimated from the measured data provide the diagonal elements of \( A_{Dopt} \).

\[ a_{Dopt} = \sigma_n^2 (i, j) \] 

Now we can calculate the diagonal elements of the weighting matrix \( V_{Dm} \).

\[ v_{Dm_0} = \mu_m^{-1} \left( \sum_{i_x} \sum_{i_y} f_m (i, j) s(i_x - j_x, i_y - j_y) \right)^2 \] 

The reciprocals determine the optimal weighting function \( w_m (i_x, i_y) \).

\[ w_m (i_x, i_y) = \mu_m \left( \frac{1}{\sum_{i_x} \sum_{i_y} f_m (i_x, i_y) s(i_x - j_x, i_y - j_y) \right)^2} \] 

Eq. 20 indicates that each optimal weighting function is matched to the corresponding regularization function in a remarkably simple way [4]. The selection of a group of suitable regu-
larization functions [4] which is not critical completely determines the optimal weighting
functions.

\[
E_Q = \sum_{i_x} \sum_{i_y} \sigma_n^{-2}(i_x, i_y) \left( g(i_x, i_y) - \sum_{j_x} \sum_{j_y} \delta(i_x, i_y, j_x, j_y) h(j_x, j_y, i_x, i_y, i_o) \right)^2 + \\
\sum_{m} \left( \sum_{i_x} \sum_{i_y} w_m(i_x, i_y) \left( \sum_{j_x} \sum_{j_y} f_m(j_x, j_y) \delta(i_x - j_x, i_y - j_y) \right)^2 \right)
\]

(21)

\[
dE_Q/d\delta(q_x, q_y) = 0 \quad \forall \quad q_x, q_y
\]

(22)

Basically the solution of the resulting linear system of equations can be found by matrix in-
version. But usually an iteration is much faster. The weighting functions lead to strong
smoothing in continuous regions and to modest smoothing at discontinuities [4].

\[
\Rightarrow \sum_{i_x} \sum_{i_y} \delta(i_x, i_y) \left( \sum_{i_x} \sum_{i_y} \sigma_n^{-2}(i_x, i_y) h(i_x, i_y, i_x, i_y, i_o) h(q_x, q_y, i_o) \right) + \\
\sum_{m} \left( \sum_{j_x} \sum_{j_y} w_m(j_x, j_y) \left( \sum_{j_x} \sum_{j_y} f_m(j_x, j_y) \delta(j_x - j_x, j_y - j_y) \right) \right) = \\
\sum_{i_x} \sum_{i_y} \sigma_n^{-2}(i_x, i_y) g(i_x, i_y) h(q_x, q_y, i_x, i_y, i_o)
\]

(23)

Unless a pure simulation is performed \(w_m(i_x, i_y)\) is not known a priori. Therefore the first guess
of the unknown signal, e.g. the result of filtered backprojection, is also used to obtain a first
estimate of \(w_m(i_x, i_y)\). If we assume that during the iteration the estimated signal converges to
the original signal, improved estimates of the weighting functions can be determined
successively from the intermediate results of the iteration. Simulation experiments showed that
even in the case of a starting guess of 0 this tricky approach converges without problems. So
we obtain an algorithm which is self updating at least up to a certain degree of accuracy [4].

Since the general matrices \(V_m\) were replaced by the diagonal matrices \(V_D\) the application
of the regularization functions to the original signal should create signals which are as "white" as
possible. On the other hand the number of coefficients of \(f_m(i_x, i_y)\) is limited. Otherwise the
number of computations required at each step of the iteration would be too large. Examples for
appropriate regularization functions are the differential operators mentioned in part 4 of this
paper. A more detailed but preliminary investigation can be found in [4].
4. Results of Simulation

Figure 2 shows the widely used Shepp and Logan head phantom [1] which provides a standard test signal for the quality assessment of CT reconstruction algorithms. It consists of a collection of ellipses of different size, brightness and orientation.

Figure 2: Test signal

Figure 3 shows the result of a reconstruction using filtered backprojection [1]. The noise which is superimposed to the projections is white and Poisson distributed. The signal to noise ratio is 30 dB. A cos²-window in the frequency domain suppresses high frequency noise which is emphasized too strongly by the filter operation before the backprojection is performed.

Figure 3: Result of filtered backprojection
Figure 4 shows a reconstruction using the adaptive algebraic approach. Since the basic element of the head phantom is an ellipse four regularization and weighting functions are used, two in direction of the coordinate axes and two in diagonal direction.

\[
\begin{align*}
  f_1(j_x, j_y) &= \delta(j_x, j_y) - \delta(j_x - 1, j_y), \\
  f_2(j_x, j_y) &= \delta(j_x, j_y) - \delta(j_x, j_y - 1), \\
  f_3(j_x, j_y) &= \delta(j_x, j_y) - \delta(j_x - 1, j_y - 1), \\
  f_4(j_x, j_y) &= \delta(j_x, j_y) - \delta(j_x + 1, j_y - 1)
\end{align*}
\]

The result of filtered backprojection is used as a starting guess for the iteration.

**Figure 4: Adaptive algebraic reconstruction**

The result of algebraic reconstruction is less grainy than that of filtered backprojection and the contours appear much clearer. On the other hand using a general purpose computer the filtered backprojection is calculated within a few seconds while the algebraic reconstruction lasts several minutes for a $100 \times 100$ pixel image. Using a special purpose hardware processor it should be possible to reduce the time for an algebraic reconstruction by several orders of magnitude. Since the algebraic reconstruction method is less sensitive to noise its main advantage seems to be the potential reduction of the required radiation dose.

5. **Conclusions**

The adaptive algebraic algorithm provides clearer reconstructions than filtered backprojection and it is less sensitive to noise. Therefore it offers the potential for a considerable reduction of the required radiation dose. The reduction of the required computer power is the main problem which remains to be solved. A special purpose reconstruction hardware should allow a reduction by several orders of magnitude.

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**References**